*An Arithmetic Treatment of Some Problems in Analysis Situs.**

BY L. D. AMES.

CONTENTS.

INTRODUCTION.

C. Jordan † has proved that the most general simple closed curve divides the plane into an interior and an exterior region. But he assumes all needed facts in regard to polygons without stating clearly just what those assumptions are. He certainly makes use of more than the special case of the same theorem for polygons. Later A. Schoenflies ‡ proved the theorem for a more restricted class of curves including polygons. But his proof is not simple. Complete arithmetization is at least impracticable; the writer must leave to the reader the last details, but these details should be only such as the reader can immediately fill in. Where the line shall be drawn must be left to the judgment

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* An abstract of the principal results of this paper was presented to the American Mathematical Society at its meeting of December, 1903, and was published in the Bulletin, March, 1904.

† *Cours d'Analyse*, 2d ed., Vol. I., §§96-103, 1893.

‡ Göttinger Nachrichten, Math.-Phys. Kl., 1896, p. 79.

of the individual writer. Schoenflies has left far more for the reader to do than have I in the proof that follows.

There is, however, one point in which Schoenflies' work is open to more serious criticism. He proves that any *straight line* which joins an interior point to an exterior point has a point in common with the curve, and then asserts in the theorem, without further consideration, that it is impossible to pass from an interior point to an exterior point without passing through a point of the curve. This does not follow. In fact it is possible to divide the points of the plane into three assemblages S_1, S_2 and B such that a point of S_1 cannot be joined to a point of S_2 by a straight line having no point in common with B, or by a curve consisting of a finite or infinite number of straight lines, but such that this can be done by other simple curves. And the essential difference between these assemblages S_1 and S_2, on the one hand, and the interior or exterior of a curve, on the other, is a property of the interior and exterior which Schoenflies leaves unmentioned in the theorem or proof, and the omission of which is our final point of criticism: namely, that each is a continuum, that is, if any point is an interior (exterior) point all points in its neighborhood are also.

More recently Ch.-J. de la Valleé Poussin * has published an outline of a proof of the same theorem for the most general simple curve. This work appears much more simple than either of the proofs already mentioned, but it is not arithmetic in form, and it is not easy to see how the arithmetization is to be effected. It can, therefore, be regarded only as a sketch of whatever rigorous proof may be made following its lines.

Since the publication of the abstract of the present paper G. A. Bliss† has proved the theorem for a somewhat more general class of curves than those for which Schoenflies proved it. O. Veblen ‡ has recently published a proof for the most general simple curve. None of these proofs deals with the corresponding theorem in three dimensions.

The present paper assumes the axioms of arithmetic but not those of geometry. It contains a proof of the above mentioned theorem for a class of curves more restricted than those of Jordan, of Valleé Poussin, or of Veblen, but more

* *Cours d'Analyse infinitesimal*, Vol. I., (1903), §§300-302.

† *The exterior and interior of a plane curve*, Bulletin of the American Mathematical Society (2), Vol. 10, (1904), p. 398.

‡ *Theory of plane curves in non-metrical analysis situs*, Transactions of the American Mathematical Society, Vol. 6, No. 1, Jan. 1905, p. 83.

general than those of Schoenflies or of Bliss. It goes back to fundamental arithmetic principles, and does not assume the theorem for the polygon. Moreover, it is extended in Part II to the corresponding theorem in three dimensions, and it seems highly probable that it could be extended to more than three dimensions. The proof, both for two and for three dimensions, is based on a conception which I have called the *order of a point* with respect to a curve [or surface]. The order is a point function, uniquely defined and constant in the neighborhood of every point not on the curve [or surface], and undefined and having a finite discontinuity at every point of the curve [or surface]. Its value is always a positive or negative integer or zero.

PART I.

IN TWO DIMENSIONAL SPACE.

I.—FUNDAMENTAL CONCEPTIONS.

1. A *point* is a complex of n real numbers $(a, b \ldots .)$. This number n is called the number of *dimensions* in which the point lies. An *assemblage* is any collection, finite or infinite, of such points. In two dimensions the assemblage of all the points is called the *plane*. In the earlier chapters we confine ourselves to two dimensions. The numbers x, y are called the *coordinates* of the point $P\ (x, y)$; the point $(0, 0)$ is called the *origin;* the assemblage of all points of the type $(x, 0)$ is called the x-axis, etc. Distance, straight lines, circles, squares, and other elementary conceptions are assumed to be defined by their usual analytic expressions without explicit mention.

2. *Transformations.* A *point transformation* is a rule by which the points of an assemblage are individually replaced by the points of an assemblage, in general different. If, whenever a property belongs to one of these assemblages it also belongs to the other, it is said to be *invariant of the transformation.* A *rigid transformation* in two dimensions is defined by relations of the type.

$$x = x' \cos \alpha - y' \sin \alpha + x_0,$$
$$y = x' \sin \alpha + y' \cos \alpha + y_0.$$

We shall assume without explicit mention the simpler facts of invariance. We shall use the expression *change of axes* for brevity to denote a rigid transformation whenever it is desired to emphasize the fact that the essential properties of the assemblage are unchanged. The reasoning involved can generally be stated in something like the following form. An assemblage is given concerning which certain facts are known. The assemblage is transformed into a second assemblage by a transformation with respect to which the given facts are known to be invariant. Certain conclusions are reached in regard to the second assemblage. This is then transformed into the given assemblage by the inverse of the first transformation. The conclusions are known to be invariant of this inverse transformation. They therefore apply to the given assemblage. Unless otherwise specified a change of axes shall be effected by a rigid transformation.

3. *Existence of a Minimum.* The following theorem is well known:*

THEOREM. If S_1 and S_2 are two complete † assemblages of points having no point in common, then the distance of any point of S_1 from any point of S_2 has a positive minimum.

4. *Curves.* A *simple curve* ‡ is an assemblage of points (x, y) which can be paired in a one to one manner with the points of the one dimensional interval $(t_0 \leqq t \leqq t_1)$ in case the curve is *not closed*, and with the points of the circle

$$\xi = \cos \lambda t, \qquad \eta = \sin \lambda t,$$

in case the curve is *closed;* moreover, when t approaches a limiting value $(\bar{t})$ the point (x, y) shall also approach a limiting point $(\bar{x}, \bar{y})$, and this limiting point shall be the point of the curve which is paired with $\bar{t}$. In the case of the open curve the points corresponding to t_0 and t_1 are called *end points.*

It follows from this definition that a *simple curve* can be represented analytically by equations of the form

$$x = \phi(t), \qquad y = \psi(t), \qquad (t_0 \leqq t \leqq t_1),$$

where $\phi(t)$ and $\psi(t)$ are single valued continuous functions, and

$$\phi(t) = \phi(t') \text{ and } \psi(t) = \psi(t')$$

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* Cf., for example, Jordan, *Cours d'Analyse*, 2d ed., Vol. I, §30, last paragraph.

† Professor Pierpont suggests complete as the English equivalent of abgeschlossen.

‡ Cf. A. Hurwitz, *Verhandlungen des ersten Internationalen Mathematiker-Kongresses*, p. 102.

are not simultaneously satisfied in the case of the open curve when $t \neq t'$, and are not simultaneously satisfied in the case of the closed curve when $t \neq t'$ except that

$$\phi(t_0) = \phi(t_1) \text{ and } \psi(t_0) = \psi(t_1).$$

If the curve is closed, let $\omega = t_1 - t_0$. It is then convenient to extend the definition of the functions $\phi(t)$ and $\psi(t)$ to all values of t by means of the relations

$$\phi(t + n\omega) = \phi(t), \; \psi(t + n\omega) = \psi(t),$$

where n is an integer and ω is defined to be the *primitive period* of the pair of functions. Conversely, every assemblage of points defined by the above equations is a simple curve.

A simple curve is said to be *smooth at a point* if the parameter can be so chosen that the first derivatives $\phi'(t)$ and $\psi'(t)$ exist, are continuous, and do not both vanish at the point. If the point is an end point one sided derivatives are admitted. A *smooth curve* is a simple curve which is smooth at every point. A *regular curve* consists of a chain of smooth curves. Analytically, it is an assemblage which can be defined by the equations

$$x = \phi(t), \qquad y = \psi(t), \qquad (t_0 \leqq t \leqq t_1),$$

where $\phi(t)$ and $\psi(t)$ are single valued continuous functions whose first derivatives $\phi'(t)$ and $\psi'(t)$ exist, are continuous and do not vanish simultaneously, except possibly at a finite number of exceptional points called *vertices*. Moreover, these derivatives approach limits as the point t approaches any such exceptional value t' from above, and also when t approaches t' from below, and in each case the limits approached by $\phi'(t)$ and $\psi'(t)$ are not both zero; the forward limits are not both equal respectively to the backward limits. It follows that one sided derivatives exist at the exceptional point and that they are equal to the respective limits.

A regular curve may admit *multiple points*, that is, points common to two or more of the constituent smooth curves, other than the common end points of two successive smooth curves. Arithmetically such points correspond to distinct values of t. Two or more of the constituent smooth curves of a regular curve may coincide along whole arcs. Such curves may be treated arithmetically in the same way as the Riemann surface is treated. We do not need such curves,

and shall include all such points without distinction under the term multiple point. Any point of a regular curve not a multiple point is a *simple point.* All of these definitions refer exclusively to assemblages of points and not to the particular way of representing them.*

The following is a special statement of a well known theorem:

THEOREM I. *A simple curve is a complete and perfect assemblage of points and lies in a finite region of the plane. Moreover, if a set of points of the curve has a limiting point* $(\bar{x}, \bar{y})$, *then the corresponding values of t have a limit which is a point of the interval* $(t_0 \leqq t \leqq t_1)$, *and* $(\bar{x}, \bar{y})$ *corresponds to* $\bar{t}$.†

The following theorem is stated without proof:

THEOREM II. *A regular curve can be divided into a finite number of parts, each of which can be represented by an equation of the form*

or else by an equation of the form

$$y = f(x)\,, \qquad (a \leqq x \leqq b)\,,$$
$$x = f(y)\,, \qquad (\alpha \leqq y \leqq \beta),$$

where f is single valued and continuous throughout the interval of definition.

Most of the results which follow apply to a somewhat more general class of curves, which by virtue of Theorem II includes all regular curves. We shall describe such a curve by saying that it satisfies the following condition:

Condition A: A curve which consists of a chain of simple curves (after the manner of a regular curve) and furthermore is such that each constituent simple curve can be represented by an equation of the form

$$y = f(x) \text{ or else by an equation of the form } x = f(y)\,,$$

where f is single valued and continuous, is said to satisfy *Condition A.*

5. *Vectors and Angles.* We define a *vector*‡ to be an object determined by the two following phenomena:

* For a discussion of change of parameter, orientation of curves, etc., see Sec. III.

† Cf. Jordan, *Cours d'Analyse*, 2d ed., Vol. 1, §§ 64, 65.

‡ This is a special case of an oriented curve, discussed in Art. 12.

(a) A simple curve which can be defined by equations of the form

$$\begin{cases} x = a_1 t + b_1, & a_1^2 + a_2^2 > 0, \\ y = a_2 t + b_2, & (t_0 \leqq t \leqq t_1); \end{cases}$$

(b) One of the two possible permutations, $P_0 P_1$ or $P_1 P_0$, of the end points.

It follows that the end points are individually invariant of any change of parameter consistent with the definition, and that a vector is also completely defined by naming the end points in a particular order, e.g. $P_0 P_1$. Taking the four points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_0'(x_0', y_0')$, and $P_1'(x_1', y_1')$, the two vectors $P_0 P_1$ and $P_0' P_1'$ are said to be *equal* if

$$x_1 - x_0 = x_1' - x_0' \quad \text{and} \quad y_1 - y_0 = y_1' - y_0'.$$

The *length* of the vector $P_0 P_1$ is the positive number

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

The *angle* θ from the vector $P_0 P_1$ to the vector $P_0' P_1'$ is defined to be any simultaneous solution of the equations

$$\sin\theta = K \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_1' - x_0' & y_1' - y_0' \end{vmatrix}, \quad \cos\theta = K \begin{vmatrix} y_1 - y_0 & -(x_1 - x_0) \\ x_1' - x_0' & y_1' - y_0' \end{vmatrix}, \qquad (A)$$

where K is the positive number

$$K = [\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\ \sqrt{(x_2' - x_0')^2 + (y_1' - y_0')^2}]^{-1}.$$

If the vectors are defined by means of their equations

$$P_0 P_1 : \begin{cases} x = a_1 t + b_1, & a_1^2 + a_2^2 > 0, \\ y = a_2 t + b_2, & (t_0 \leqq t \leqq t_1), \end{cases}$$

$$P_0' P_1' : \begin{cases} x = a_1' t + b_1', & a_1'^2 + a_2'^2 > 0, \\ y = a_2' t + b_2', & (t_0' \leqq t' \leqq t_1'), \end{cases}$$

where the parameter t is so chosen that the value of t at the first named end point is less than that at the last named end point, then equations (A) are equivalent to the equations

$$\sin\theta = \varkappa \begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix}, \quad \cos\theta = \varkappa \begin{vmatrix} a_2 & -a_1 \\ a_1' & a_2' \end{vmatrix}, \qquad (B)$$

where K is the positive number

$$\varkappa = [\sqrt{a_1^2 + a_2^2}\quad \sqrt{a_1'^2 + a_2'^2}]^{-1}.$$

We here assume an analytic definition of the trigonometric functions.* Ordinarily we select a particular solution by some convention. The angle ABC shall mean the angle from BA to BC.

To justify the above definition it can be shown:

(*a*) That equations (A), and hence (B), always have solutions differing by multiples of 2π;

(*b*) That if the angle from $P_0 P_1$ *to* $P_0' P_1'$ is θ, and that from $P_0'P_1'$ *to* $P_0''P_1''$ is θ', then the angle from P_0P_1 *to* $P_0''P_1''$ is $\theta + \theta' + 2n\pi$, where n is a positive or negative integer or zero;

(*c*) That θ is invariant of any rigid transformation.

6. *Continua.* A *two-dimensional continuum* is an assemblage of points $P(x, y)$ such that:

(*a*) If P_0 (x_0, y_0) is a point of the assemblage, all points in the two dimensional neighborhood:

$$| x - x_0 | < h, \qquad | y - y_0 | < h$$

of P_0 belong to the assemblage;

(*b*) Any two points of the assemblage can be joined by a simple curve consisting wholly of points of the assemblage.

The *neighborhood* of a point P_0 may be defined generally to be a continuum containing P_0 and such that the distance of any of its points from P_0 is less than h, where h is a positive constant as small as either party to a discussion wishes. The term *near* will be used as a technical term to replace the longer and more familiar expression *in the neighborhood of.*

Any point of a continuum is an *interior point.* A *boundary point* of a continuum is a point not belonging to the continuum but having points of the continuum in its neighborhood. Any point P_0 not an interior or boundary point is an *exterior point*, and all points near P_0 are exterior points.

The term *region* is applied both to a continuum, and to a continuum plus its boundary. A region is said to be *finite* if it is possible to choose a constant G so that if $P(x, y)$ is any point of the region, then

$$| x | + | y | < G.$$

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* Cf. for instance, Godefroy, *Théorie élémentaire des Séries*, Chap. 5.

The boundary of a region is a complete assemblage.* If two continua have a point in common, then the totality of the points of the two continua taken together form one continuum. The continuum S' is said to be *annexed* to the continuum S along B (a part or all of their common boundary) if the points of S, S', and B form one continuum and are so considered. The definitions of this section are invariant of any one-to-one and continuous transformation.

There are at least two essentially distinct ways of defining a particular continuum. One is by defining the points of the continuum explicitly. The more common way is by defining the boundary explicitly. Section II deals with theorems relating to the second method. The following are examples of the first method.

Example 1. The interior of a circle may be defined by the inequality

$$x^2 + y^2 - r^2 < 0.$$

Example 2. The interior of a triangle, or more generally the assemblage S, defined by relations of the form

$$u_i(x, y) \equiv A_i x + B_i y + C_i > 0, \qquad (i = 1, 2, 3),$$

where these equations are satisfied by at least one point, can be proved to be a continuum as follows:

(*a*) Let P_0 (x_0, y_0) be one point of S. Since u_i is continuous and $u_i(x_0, y_0) > 0$, hence if P (x, y) is any point near P_0, $u_i(x, y) > 0$, and hence P belongs to S.

(*b*) Let P_0 (x_0, y_0) and P_1 (x_1, y_1) be any two points of S. Hence

$$\begin{aligned} u_i(x_0, y_0) &\equiv A_i x_0 + B_i y_0 + C_i > 0, \\ u_i(x_1, y_1) &\equiv A_i x_1 + B_i y_1 + C_i > 0, \end{aligned} \qquad (i = 1, 2, 3),$$

and therefore

$$A_i\left(\frac{\lambda_1 x_0 + \lambda_2 x_1}{\lambda_1 + \lambda_2}\right) + B_i\left(\frac{\lambda_1 y_0 + \lambda_2 y_1}{\lambda_1 + \lambda_2}\right) + C_i > 0, \quad (i = 1, 2, 3),$$

where λ_1 and λ_2 are any numbers not negative and not both zero. This is a sufficient condition that the point

$$\left(\frac{\lambda_1 x_0 + \lambda_2 x_1}{\lambda_1 + \lambda_2}, \frac{\lambda_1 y_0 + \lambda_2 y_1}{\lambda_1 + \lambda_2}\right)$$

* Cf. Jordan, ibid., §23.

belongs to S. But any point of the segment $P_0 P_1$ can be expressed in this form. Hence any two points of S can be joined by a straight line wholly in S. Hence S is a continuum.

Example 3. Let an assemblage S (Fig. 1) of points $P(x, y)$ be defined by the relations

$$x_0 < x < x_1, \qquad y = f(x) + r, \qquad 0 < r < h, \tag{1}$$

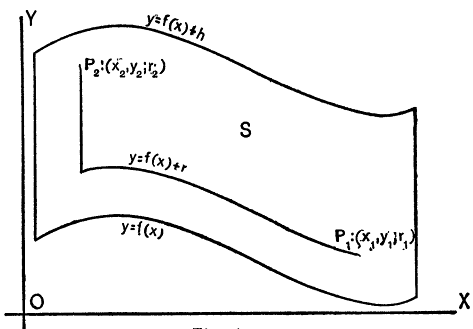

Fig. 1

where $f(x)$ is single valued and continuous, and h is constant. We proceed to prove that S is a continuum.

(a) Let $\bar{P}(\bar{x}, \bar{y}, \bar{r})$ be any point of S. Choose $\delta > 0$ so that

$$2\delta < \bar{r} \quad \text{and} \quad 2\delta < h - \bar{r};$$

that is

$$2\delta < \bar{r} < h - 2\delta. \tag{2}$$

Since $f(x)$ is continuous it is possible to choose $\varepsilon > 0$ so that

$$f(x) - \delta < f(\bar{x}) < f(x) + \delta \tag{3}$$

when

$$\bar{x} - \varepsilon < x < \bar{x} + \varepsilon,$$

and so that

$$x_0 < \bar{x} - \varepsilon \quad \text{and} \quad \bar{x} + \varepsilon < x_1. \tag{4}$$

We now proceed to show that any point $P(x, y)$ in the neighborhood

$$\bar{x} - \varepsilon < x < \bar{x} + \varepsilon, \tag{5}$$

$$\bar{y} - \delta < y < \bar{y} + \delta \tag{6}$$

of $\bar{P}$ lies in S. From (4) and (5)

$$x_0 < x < x_1. \tag{7}$$

Adding (6), (2), and (3) and simplifying by means of (1) we obtain

$$f(x) < \qquad y \qquad < f(x) + h, \tag{8}$$

that is, $$y = f(x) + r, \qquad \text{where } 0 < r < h. \tag{9}$$

But (7) and (9) are the condition that P is in S,

(*b*) Let $P_1(x_1, y_1;\ r_1)$ and $P_2(x_2, y_2;\ r_2)$ be any two points in S, and let $r_1, \leqq r_2$. They can be joined by a simple curve in S defined as follows:

$$x = x_2, \qquad f(x) + r_1 \leqq y \leqq f(x) + r_2,$$

and $$y = f(x) + r_1, \qquad x_1 \leqq x \leqq x_2 \text{ or } x_2 \leqq x \leqq x_1.$$

Hence S is a continuum.

II. THE THEOREM RELATING TO THE DIVISION OF THE PLANE BY A SIMPLE CLOSED CURVE.

7. *Order of a Point.* Given any closed curve whose equations are

$$x = \phi(t), \qquad y = \psi(t),$$

where ω is the primitive period of the pair of functions $\phi(t)$ and $\psi(t)$. Let $P \ (t)$ be a variable point on the curve, and O a fixed point not on the curve. Let $\theta(t)$ be the angle which OP makes with the positive x-axis, or its equivalent for this purpose, the vector $(0, 0)(1, 0)$. Then θ is an infinitely multiple valued function of t for all values of t, such that any two values of θ corresponding to the same value of t differ by a multiple of 2π. Let t_0 be a particular value of t and let $\theta_i(t_0)$ be a particular one of the values of $\theta(t_0)$. Then it is possible to choose from the different values of $\theta(t)$ one and only one set of values which form a single valued continuous function of t taking on the chosen value $\theta_i(t_0)$ when $t = t_0$.* Call this single-valued function $\theta(t)$. The values of t and $t + \omega$ represent the same point, and the values of the multiple valued $\theta(t)$ at this point differ by a multiple of 2π. Hence

$$\theta(t + \omega) = \theta(t) + 2n\pi,$$

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* Cf. for example, Stoltz, *Differential Rechnung*, Vol. 2, p. 15-20.

where n is a positive or negative integer or zero. Then n is defined to be the *order* of the point O with respect to the particular parametric representation of the curve.* The order is not defined for any point on the curve.

That the order depends only on O and not on the particular value of t chosen may be seen as follows: Let t vary continuously. Then $\theta(t)$ and $\theta(t+\omega)$, and hence n vary continuously. But n can vary only by integers. Hence n is constant. That n is independent of the particular one of the possible single valued functions chosen is seen in a similar manner. That n is invariant of a rigid transformation follows from the fact that $\theta(t)$ and $\theta(t+\omega)$ are invariant.

8. We proceed to prove some theorems about the order of a point.

THEOREM I. *If a point is of order n with respect to a given closed curve, then all points near it are of order n.*

Proof. Let O_1 be a point not on the curve, and O any point near it. Let θ_1 and θ be the angles which O_1P and OP respectively make with the x-axis. Then if O_1O is sufficiently small, $|\theta_1 - \theta|$ is less than an arbitrarily preassigned number for all points on the curve. In particular

$$[\theta_1(t+\omega) - \theta_1(t)] - [\theta(t+\omega) - \theta(t)] < 2\pi.$$

Hence the order of O differs from that of O_1 by less than unity. But both are integers. Hence they are equal.

Corollary. The points of the order of a given point form one or more continua.

THEOREM II. *If two points are of different orders with respect to a given closed simple curve C, any simple curve joining them has a point in common with C.*

Proof. Let the end points $P_0(t_0)$ and $P_1(t_1)$, $(t_0 < t_1)$, of a simple curve C' be of orders m and n respectively with regard to the closed curve C. Consider the upper limit $\bar{t}$ of the values of t corresponding to points of order m. Then there are points of order m and other points not of order m near $\bar{P}(\bar{t})$. If $\bar{P}$ is not on the curve C this contradicts Theorem I. Hence, $\bar{P}$ is on the curve C.

*It is sufficient for the present chapter to consider only one particular parametric representation. See Art. 10 for a discussion of the invariance of n with respect to a change of parameter.

9. The theorem relating to the division of the plane by a closed simple curve will be proved by the aid of two lemmas corresponding to the statements that the curve divides the plane into *at least two*, and *at most two* continua.

FIRST LEMMA. *Near any point P_0 of a simple closed curve which satisfies Condition A there are two points of orders differing by unity.*

Proof. The curve consists of a finite number of parts, each of which can be represented by an equation of the form

$$(a) \quad y = f(x),$$

or else by an equation of the form $(b) \quad x = f(y)$,

where f is single valued and continuous. If P_0 is an end point of one of these parts, then there is a point near P_0 which is not such an end point. Hence we may assume without loss of generality that P_0 is not such an end point. Suppose that the part on which P_0 lies can be represented by the equation

$$y = f(x).$$

The other case is similar. Transform to new axes parallel to the original axes and having P_0 as origin (Fig. 2). All the conditions are invariant of this transformation.

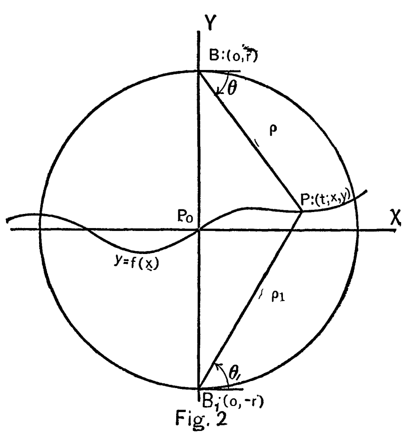

Fig. 2

The y-axis has no point other than P_0 in common with the curve near P_0. Hence it is possible to choose R so small that if $r \leqq R$, and B is the point (o, r), and B_1 the point $(o, -r)$, the segment BB_1 has no point on the curve except P_0. Let $P(t; x, y)$ be any point on the curve. Let θ and θ_1 be the

angles BP and $B_1 P$ respectively make with the positive x-axis. Let $\rho = \overline{BP}$ and $\rho_1 = \overline{B_1 P}$, where ρ, ρ_1, and r are positive numbers. By the definition of an angle,

$$\sin\theta = \frac{y-r}{\rho}, \qquad \cos\theta = \frac{x}{\rho}$$
$$\sin\theta_1 = \frac{y+r}{\rho_1}, \qquad \cos\theta_1 = \frac{x}{\rho_1}.$$

Let $\phi = \theta - \theta_1$. Then

$$\sin\phi = \frac{-2rx}{\rho\rho_1}, \qquad \cos\phi = \frac{(x^2+y^2)-r^2}{\rho\rho_1}.$$

From these relations, if ω is the primitive period and ε sufficiently small, and n, n_1, and n_2 integers, positive, negative, or zero, and x increases as t increases near P_0, it follows that

$$\begin{aligned}
\phi(t_0) &= (2n_1+1)\pi, \\
\phi(t_0+\varepsilon) &> (2n_1+1)\pi, \\
\phi(t_0+\omega-\varepsilon) &< (2n_2+1)\pi, \\
\phi(t_0+\omega) &= (2n_2+1)\pi.
\end{aligned}$$

When $t_0 < t < t_0 + \omega$, $\sin\phi$ can vanish only when $x = 0$, and in this case $(x^2+y^2) - r^2$, and hence $\cos\phi$ is positive. Hence

$$\phi(t) \neq (2n+1)\pi \qquad \text{when} \qquad t_0 < t < t_0 + \omega,$$

and therefore $n_2 - n_1 = 1$. Hence

$$[\theta(t_0+\omega) - \theta(t_0)] - [\theta'(t_0+\omega) - \theta'(t_0)] = \phi(t_0+\omega) - \phi(t_0) = 2\pi.$$

Hence the order of B exceeds that of B' by unity.

SECOND LEMMA. *Given the continuum R, and the curve AB:*

$$y = f(x), \qquad \text{or} \qquad x = f(y),$$

where f is single valued and continuous:

(a) *If R contains all points of the curve, except possibly its end points, which may lie in the boundary of R, then the totality R^- of points of R not on AB form at most two continua;*

(b) *If also one or both end points lie in R, then R^- is one continuum.*

Proof. (a) Suppose the curve can be represented by the equation

$$y = f(x), \qquad \text{(Fig. 3).}$$

The other case is entirely similar. Draw a straight line CD parallel to the y-axis, lying wholly in R, and bisected at a point of the curve AB, with C lying above the curve. Let P be any point of R^- which cannot be joined to D by a simple curve wholly in R^-. If there is no such point the theorem is granted.

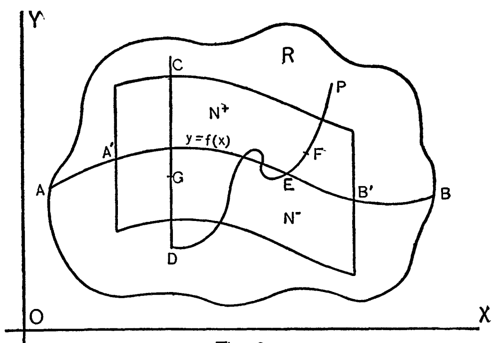

Fig. 3

Otherwise join P to D by a simple curve PD wholly in R. This curve will have a point in common with the curve AB. Let PE be an arc of PD having one extremity E on the curve AB, but containing no other point of this curve, Choose an arc $A'B'$ of AB which contains E and also the point common to AB and CD in its interior, but does not contain A or B. Along this arc construct two continua like that of Art. 6, Example 3, one above, and one below $A'B'$, lying wholly in R^-, and denote them by N^+ and N^- respectively. Choose a point F on PE so near to E that it lies either in N^+ or N^-. Suppose it lay in N^-. Choose a point G on CD in N^-. Then F and G can be joined by a simple curve wholly in N^-. Hence the simple curve $PFGD$ lies wholly in R^-. which is contrary to hypothesis. Hence F must lie in N^+, and by similar reasoning P can be joined to A by a simple curve wholly in R^-. Hence the points of R^- form *at most two* continua.

(b) Suppose the curve is represented as in the first case but let B (Fig. 4) lie in R. Extend the curve AB slightly parallel to the x-axis, to B'. By the first case the points of R not on AB' form at most two continua. If they form

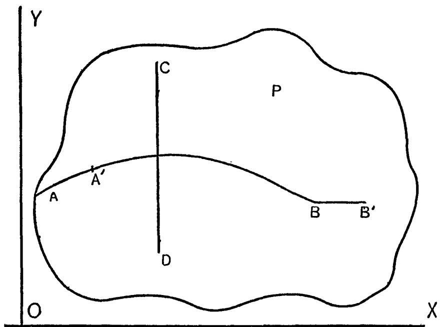

Fig. 4

one continuum the theorem is granted. If they form two continua, by the adjunction of the points of BB' exclusive of B these can be annexed to each other, thus forming one continuum.

MAIN THEOREM. *The points of the plane not on a given simple closed plane curve satisfying Condition A form two continua of each of which the entire curve is the total boundary.*

Proof. In the neighborhood of any point of the curve there are two points of different orders with respect to the curve (First Lemma). Hence the points of the plane not on the curve form *at least two* continua (Art. 8, Th. I and Cor., Th. II, also Art. 6). Divide the curve into a finite number of parts each of which can be represented by an equation of the form $y = f(x)$, or else by an equation of the form $x = f(y)$. Construct these in the order in which they appear in the curve. By the second part of the Second Lemma each of these except the last does not divide the region consisting of the plane less the points already cut out. The last divides the plane into *at most two* continua. Hence the points of the plane not on the curve form just two continua.

Any point of the curve is a boundary point of each continuum (First Lemma, and Arts. 8 and 6). Any point not on the curve belongs to one of the continua, and hence cannot be a boundary point.

III. REGIONS, ORIENTATION OF CURVES, NORMALS, AND RELATED TOPICS.

10. *Change of Parameter.* The following theorem is stated without proof:

THEOREM. *Let a given simple curve be defined by two sets of equations*

$$\begin{cases} x = \phi(t), \\ y = \psi(t), \end{cases} \quad t_0 \leqq t \leqq t_1, \quad \text{and} \quad \begin{cases} x = \bar{\phi}(t), \\ y = \bar{\psi}(t), \end{cases} \quad t'_0 \leqq t' \leqq t'_1,$$

where ϕ, ψ, $\bar{\phi}$ and $\bar{\psi}$ are single valued, continuous functions.

(a) *In the case of the open curve, if no two values of $t\,[t']$ yield the same point, and if the values of t and t' which yield the same point of the curve are assigned to each other, then t is a single valued, continuous function $f(t')$, monotonic and never constant throughout the interval of definition; and the same two points are given as the end points in each case;*

(b) *In the case of the closed curve, if no two values of $t\,[t']$ yield the same point unless they differ by a period of the pair of functions, the values of t and t' which yield the same point of the curve can be assigned to each other in such a way that $t' = f(t)$, where $f(t)$ is single valued and continuous for all values of t, monotonic and never constant.*

The totality of transformations $t' = f(t)$ thus defined form a group G. Such a transformation is said to be *even* if an increase in t yields an increase in t'. The even transformations of G form a subgroup G^+ of G. Any transformation of G is an even transformation or is equivalent to an even transformation followed by the transformation $t' = -t$. The order n of a point with respect to the curve is invariant of any even transformation. If t is replaced by $-t'$ the sign of the order of a point is reversed. Then n^2 is invariant of any transformation of G.

11. *Interior and Exterior.*

THEOREM. *All sufficiently distant points are of order zero with respect to a given closed curve.*

Proof. Let $P_1 \quad (x_1, y_1)$ be a distant point, and let $P \quad (x, y)$ be a variable point on the curve. Let θ be the angle P_1P makes with the positive x-axis. Then by Art. 5,

$$\cos\theta = (x - x_1)/P_1P, \qquad \sin\theta = (y - y_1)/P_1P.$$

Then if $\sqrt{x_1^2 + y_1^2}$ is taken sufficiently large either $\cos\theta$ or $\sin\theta$ never changes its sign as P varies. In either case the maximum variation of θ is less than π. Hence the order of P_1 is zero.

If the points of a continuum are all of order n the *continuum* is defined to be of *order* n. The *exterior* of a simple closed curve is defined to be that one of the two continua into which the curve divides the plane which contains all sufficiently distant points. The other continuum is defined to be the *interior*. It follows that the exterior is of order zero, and the interior of order ± 1. If the interior is of order -1, the parameter can be so chosen that the order of the interior will be $+1$. The *neighborhood of a curve* is a continuum containing all points of the curve, and such that if P is a point of the continuum, and P_1 a suitably chosen point of the curve, then $PP_1 < h$, where h is a positive constant previously chosen as small as either party to a discussion wishes.

We have proved incidentally the following theorem, which for greater clearness we state somewhat freely in geometric language.

THEOREM. *Let P be a variable point on a simple closed regular curve, and A any fixed point not on the curve. Then when P traces the curve and returns to its initial position, the angle which AP makes with the positive x-axis, varying continuously returns to its initial value if A is an exterior point of the curve, and is changed by 2π if A is an interior point.*

12. *Orientation of Curves.* The conception of an *oriented* curve is a generalization of that of a vector. It is often desirable to distinguish the positive from the negative sense along a curve. The process or the result of making this distinction we will call *orientation.* More explicitly, we define an *oriented curve* and then define the *positive sense along* such a curve. An *oriented* simple curve is defined to be an object determined by the two following phenomena:

(a) A simple curve;

(b) One of the two possible permutations AB or BA of the end points of any one open arc of the curve.

If the orientation of a given simple curve is defined by the permutation P_1P_2 of the end points $P_1(t_1)$ and $P_2(t_2)$ of a definite open arc, and $P(t)$ is any point of that arc, then we will agree to choose the parameter so that $t_1 \leqq t \leqq t_2$, and conversely. If a change of parameter is necessary to effect this it can always be accomplished by the transformation $t = -t'$. Thus a permutation of the end

points of any open arc is uniquely determined. Hence a simple curve can be oriented in two and only two distinct ways, and one of these is fully determined by a permutation of the end points of an arbitrary arc. With this agreement as to the choice of parameter, a point is said to *trace* a simple curve in the *positive sense* if its parameter *increases* continuously. A line integral is said to be *extended along* the arc $t_1\, t_2$ in the *positive sense* if t_1 is the lower limit and t_2 the upper limit of integration. The sign of the order of a point with respect to a curve is reversed by reversing the orientation of the curve.

In general the orientation of an open curve is entirely arbitrary. If a simple curve is considered as part or all of the boundary of a definite region, we will agree that the curve shall be so oriented that the region shall be of order one greater than the region from which the curve separates it (see Fig. 5). If a

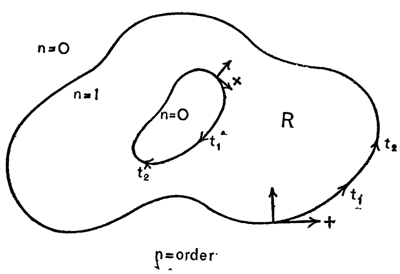

Fig. 5

closed curve is not explicitly considered as a boundary of a region exterior to it, it shall be oriented so that its interior is of order one greater than its exterior, in other words, so that if O is an interior point and P traces the curve in the positive sense returning to its initial position, the angle which OP makes with a fixed line varying continuously shall be *increased* by 2π (see Art. 11).

If a one-to-one relation is established between the points of two simple curves and the parameters have been chosen as above, then they are said to have the *same orientation*, or to be *similarly oriented* if the parameter of one is an increasing function of that of the other. They are said to have *opposite orientations*, or to be *oppositely oriented* if the parameter of one is a decreasing function of that of the other. A special case of this is that in which two curves coincide along a given arc. In this case coincident points in the two curves are assigned to each other, unless otherwise specified.

THEOREM. *Let two plane regions R_1 and R_2 each form the interior of a simple closed curve C_1 and C_2 respectively, satisfying Condition A. Let a segment σ_1 of C_1 coincide with a segment σ_2 of C_2, then*

(a) *If R_1 and R_2 are exterior to each other, the orientation of σ_1 is opposite to that of σ_2;*

(b) *If R_1 is wholly interior to R_2, the orientation of σ_1 is the same as that of σ_2.*

Proof (a). Choose a part or all of σ_1 (or σ_2) which can be represented in the form

(1) $y = f(x)$, or else in the form (2) $x = f(y)$,

where f is a single valued, continuous function. Let P_0 be any point of this segment not an end point. The orientation of σ_1 can not be the same as that of σ_2, for suppose it were. Then by Art. 9, First Lemma, near P_0 there are two points B and B' such that the order of B with respect to either curve is greater than that of B' by unity. Hence by Art. 11 B is interior to each curve, which is contrary to hypothesis. The second case is proved similarly.

The property of the plane stated in this theorem is later taken as the definition of a bilateral surface. It is not true on a unilateral surface. Thus it will follow that the plane is bilateral. Goursat tacitly assumes this theorem or an equivalent one in his proof of Cauchy's Integral Theorem.* It is assumed whenever an integral taken around any region R is assumed to be equal to the sum of the integrals taken around the mutually exclusive regions of which R consists, and in analogous cases involving variation, or analytic continuation along closed paths in the study of multiple valued functions.

13. *Tangents and Normals.* If a simple closed regular curve is represented by the equations

$$x = \phi(t), \qquad y = \psi(t)$$

and if its orientation has been defined, and the parameter has been chosen according to the specifications of Art. 12, then the *positive tangent* at a point P_0 (t_0) at which the curve is smooth is the vector P_0 (s_0) P_1 (s_1) defined by the equations

$$\begin{aligned} x &= s\phi'(t_0) + \phi(t_0), \\ y &= s\psi'(t_0) + \psi(t_0), \end{aligned} \qquad (s_0 = 0 \leqq s \leqq s_1).$$

* Acta Mathematica, *t*. IV, p. 197. Or see Harkness and Morley, *Theory of Functions* (1893), p. 164

A *normal* to the curve at a point P_0 (t_0) at which the curve is smooth is a vector defined by the equations

$$\begin{aligned} x &= -\varepsilon s\phi'(t_0) + \phi(t_0), && (s_0 = 0 \leqq s \leqq s_1), \\ y &= \varepsilon s\psi'(t_0) + \psi(t_0), && (\varepsilon = \pm 1). \end{aligned}$$

If this enters the interior of the curve in the neighborhood of the curve it is called the *inner normal.* It follows from the proof of the first lemma, Art. 9, that in the case of the inner normal $\varepsilon = +1$, and the inner normal makes an angle of $+\pi/2$ with the positive tangent. Even when the region is exterior to one of its boundaries, these conclusions are equally true of that normal which enters the interior of the region provided the orientation of the boundary is chosen according to the specifications of Art. 12 (see Fig. 5).

14. *Regions and Boundaries.* As an illustration of a class of theorems which are often assumed without even mention, but which are by no means trivial, the following theorems are stated, mostly without proof. A *loop-cut* is defined to be a simple closed curve lying wholly in a continuum under consideration.

THEOREM I. *The totality R^- of the points of a plane continuum R not on a given loop-cut L satisfying Condition A form two continua, one of which is wholly interior and one wholly exterior to the loop-cut, and every point of the loop cut is a boundary point of each.*

The proof is similar to that of the main theorem of Art. 9, which is a special case of this.

THEOREM II. *If a loop-cut L is drawn in the interior of a closed simple curve C, each satisfying Condition A:*

(a) *The interior of L lies wholly interior to C, and is wholly bounded by L;*

(b) *The exterior and perimeter of C lies wholly exterior to L, and the exterior of C is bounded wholly by C;*

(c) *There exist points exterior to L and interior to C, and they form a continuum of which C and L form the total boundary.*

Proof. The main points of the proof may be exhibited in outline as follows:

Each point of the plane belongs to one of nine mutually exclusive classes:

	C_i	C	C_e
L_i		*No point.* (3)	*No point.* (2)
L		*No point.* (1)	*No point.* (1)
L_e			

where C_i and C_e denote the interior and exterior respectively of C, etc.

(1) By hypothesis, no point of L belongs to C or to C_e.

(2) No point is in C_e and L_i. Suppose P were in C_e and L_i. Choose a distant point A. This lies in C_e and in L_e. Hence A and P can be joined by a curve wholly in C_e. This curve must cut L. Hence a point of L is in C_e. This contradicts (1).

(3) Suppose a point P of C were in L_i, then all points near P are in L_i. But there are points of C_e near P. This contradicts (2).

By Theorem I the points of C_i consist of the curve L, and two continua belonging to L_i and L_e respectively. From the diagram it is seen that C is wholly in L_e. Hence by Theorem I the points of L_e consist of the curve C and two continua belonging to C_i and C_e respectively. Hence there are points of each class left blank in the diagram. The first clause of (a), (b), and (c) can now be read off from the diagram. The remainder of the proof is left to the reader.

THEOREM III. *If two simple closed curves C and L each satisfying Condition A are wholly exterior to each other:*

(a) *The interior of C is wholly exterior to L, and is wholly bounded by C, and similarly interchanging letters;*

(b) *There exist points exterior to each, and these form a continuum bounded wholly by C and L.*

The proof is similar to that of Theorem II.

THEOREM IV. *If n simple closed curves satisfying Condition A have no point in common:*

(*a*) *A necessary and sufficient condition that they form the total boundary of an infinite region is that they lie wholly exterior to each other; the region so bounded is exterior to each;*

(*b*) *A necessary and sufficient condition that they form the total boundary of a finite region is that $n - 1$ of the curves are wholly interior to the remaining one and wholly exterior to each other; the region so bounded is exterior to each of the $n - 1$ curves and interior to the remaining one.*

This can be proved by mathematical induction.

PART II.

IN THREE DIMENSIONAL SPACE.

I.—FUNDAMENTAL CONCEPTIONS.

15. Some of the fundamental conceptions made use of in the following chapters have been discussed in the introduction for space of two dimensions. Those definitions and principles will now be extended to space of three dimensions by the addition of a third variable, without further comment, whenever no difficulty presents itself in so doing. We shall prove the theorem that a simple closed surface divides space into two continua, first for a very restricted class of surfaces, and later indicate how the proof can be extended to more general cases. The proof follows a method similar to that used in two dimensions. A preliminary discussion of certain fundamental conceptions is necessary.

16. *Surfaces.* *A smooth simple closed surface* is an assemblage of points $P(x, y, z)$ defined as follows:

(*a*) If $P_0(x_0, y_0, z_0)$ is a point of the assemblage, it is possible to choose three equations

$$x = \phi(u, v), \qquad y = \psi(u, v), \qquad z = \chi(u, v), \tag{A}$$

where

$$x_0 = \phi(u_0, v_0), \quad y_0 = \psi(u_0, v_0), \quad z_0 = \chi(u_0, v_0),$$

where ϕ, ψ, χ are single valued near (u_0, v_0), where all points given by these equations near (u_0, v_0) are points of the assemblage, and where no point (x, y, z) is given by two distinct points (u, v) near (u_0, v_0).

(*b*) The assemblage is *simple*, that is, if P_0 (x_0, y_0, z_0) is a point of the assemblage, all points in the three dimensional neighborhood of P_0 can be given by one set of parametric equations (A) as just defined.

(*c*) The assemblage is *complete*, that is, if $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ is a limiting point of the assemblage, then it shall belong to the assemblage.

(*d*) The assemblage is *connected*, that is, if $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are any two points of the assemblage, then it is possible to draw a simple curve

$$x = \lambda(t), \qquad y = \mu(t), \qquad z = \nu(t), \qquad (t_0 \leqq t \leqq t_1),$$

having P_0 and P_1 as end points and such that all points of the curve are points of the assemblage.

(*e*) The assemblage lies in a *finite region*, that is, it is possible to choose a constant G so that if $P(x, y, z)$ is a point of the assemblage, then

$$|x| + |y| + |z| < G.$$

(*f*) The assemblage is *smooth at every point*, or simply *smooth*, that is, if P_0 (x_0, y_0, z_0) is a point of the assemblage, it is possible to choose the equations (A) so that the first partial derivatives

$$\phi_u \left(= \frac{\partial \phi}{\partial u}\right), \phi_v, \psi_u, \psi_v, \chi_u, \chi_v$$

are single valued and continuous near $(u_0, v_0,)$, and so that the Jacobians

$$J_x = \begin{vmatrix} \psi_u & \chi_u \\ \psi_v & \chi_v \end{vmatrix}, \quad J_y = \begin{vmatrix} \psi_u & \phi_u \\ \chi_v & \phi_v \end{vmatrix}, \quad J_z = \begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}$$

do not all vanish at P_0.

By virtue of (*a*), (*c*) and (*e*) the assemblage is said to be *closed*.

17. *Dissection of Surfaces.*

THEOREM I. *A smooth simple closed surface can be divided into a finite number of parts, each of which has the following properties:*

(*a*) *It can be represented by an equation of one of the following forms:*

$$x = f(y, z), \quad \text{or} \quad y = f(z, x), \quad \text{or} \quad z = f(x, y),$$

where f and its first partial derivatives are single-valued and continuous;

(*b*) *It is bounded by one simple regular curve;*

(*c*) *It can be included in a sphere of arbitrarily preassigned radius.*

To prove this we divide space into cubes of edge δ. If δ is chosen sufficiently small, either the part of the surface in any cube consists of a finite number of prices each of which satisfies the requirements of the theorem, or, in case the surface is tangent to a face, the part of the surface in the two cubes having this face in common satisfies the requirements of the theorem. The details of the arithmetization are omitted.

Most of the discussions which follow apply to any simple surface which can be dissected as specified in Theorem I. We shall describe such a surface by saying that it satisfies the following condition:

Condition B: A surface is said to satisfy *Condition B* if it consists of a finite number of parts each of which answers to the description in Theorem I, and if, moreover, the surface satisfies conditions (b, c, d, e) of Art. 15. If it also satisfies (a) it is said to be *closed.*

18. *Parametric Representation of Surfaces.* The following theorems relate to the possibility of representing a given surface by two different systems of parametric equations, and to the relations of the two parametric planes. The transformations involved are analogous to the transformation $t = f(t')$ by which the parameter t of a simple curve is replaced by a different parameter t', where $f(t')$ is monotonic and never constant (Art. 10). The theorems are given without proof.

THEOREM II. *If two planes R and R' are mapped in a one to one and continuous manner on each other, and a simple closed curve C in R is mapped on a simple closed curve C' in R', each of which is oriented, then*

(*a*) *Any interior (exterior) point of C is mapped on an interior (exterior) point of C';*

(*b*) *All such transformations form a group G; of these there are transformations, called* EVEN *transformations, for which the curves of every such pair have the same orientation, and these transformations form a subgroup G^+ of G; there are transformations called* ODD *transformations for which the curves of every such pair have opposite orientations, in particular the transformation $x' = x$, $y' = -y$; the totality of even and odd transformations exhaust G; moreover, the group of odd transformations can be generated by the particular odd transformation $x' = x$, $y' = -y$ in succession with each of the even transformations;*

(*c*) *If the Jacobian of such a transformation is defined and continuous at every point of the regions involved, and does not vanish in them, then the necessary and sufficient condition that the transformation is even is that the Jacobian is positive at every point.*

THEOREM III. *If a finite simple surface region R including its boundary, which is a simple closed curve C, is mapped in a one to one and continuous manner on a portion R' of a plane, then R' forms the interior and boundary of the closed curve C' on which C is mapped.*

19. *Unilateral and Bilateral Surfaces. Orientation.* Given any simple surface. Let R_i be any complete open region of the surface, and let C_i be a simple closed curve forming part or all of its boundary. If, having oriented one such boundary, it is then possible in one and only one way to orient every such boundary so that if C_i and C_j are any two of these having a common segment σ,

(*a*) C_i and C_j shall be oppositely oriented along σ when R_i and R_j are exterior to each other, and

(*b*) C_i and C_j shall be similarly oriented when either R_i or R_j is interior to the other,

then the surface is said to be *bilateral.* If this is not possible the surface is said to be *unilateral.* We will agree that if one such curve on a bilateral surface is oriented, then the orientation of every such curve on the surface shall be consistent with the above specifications.

Let us now extend to surfaces the idea of orientation. We define an *oriented simple surface* to be an object determined by the three following phenomena:

(*a*) A simple bilateral surface;

(*b*) A definite complete open region of the surface;

(*c*) An oriented simple curve forming part or all of the boundary of that region.

It follows that a simple bilateral surface can be oriented in two and only two distinct ways. If a simple bilateral surface is oriented and is represented by three equations of the form

$$x = \phi(u, v), \qquad y = \psi(u, v), \qquad z = \chi(u, v),$$

and if a complete open region R_i of the surface, bounded by C_i, corresponds to the region R_i' bounded by C_i' of the uv-plane, then if C_i and C_i' are not similarly oriented they will be after the substitution $u = -u'$, $v = v'$. We shall assume that the parameter has been so chosen. Then by the theorem of Art. 12 and Theorem III of Art. 18 the same is true for every such curve. Thus the different parts of the surface may be given by different analytic representations and the validity and definiteness of our definitions be not affected.

II. Solid Angles and Order of a Point.

20. *Solid Angles.* Let us start from the ordinary conception of a solid angle. Define a system of spherical coordinates by the relations

$$\begin{aligned} x &= \rho \sin \phi \cos \theta, \\ y &= \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi. \end{aligned}$$

We shall speak of the line $\phi = 0$ and $\phi = \pi$ as the *polar axis.* Let a piece R of a surface be defined by the equation

$$\rho = f(\theta, \phi),$$

where $f(\theta, \phi)$ is a single valued function of θ and ϕ throughout a region R' of the surface of the unit sphere. Then, according to the ordinary conception, the solid angle subtended by R at the origin is the area of R', which is given by the integral

$$\int\int_{R'} \sin\phi \, d\theta \, d\phi.$$

We wish to extend this definition in such a way that the solid angle shall be susceptible of an algebraic sign. To illustrate our purpose, consider the convex surface of a circular cylinder so situated that the origin is exterior to it, and so that it is not pierced by the polar axis, and so that some of the radii vectores cut it in two points. It can be divided into two parts so that each can be represented by one equation of the form $\rho = f(\theta, \phi)$. We wish to define the solid angle subtended by the cylindrical surface so that the contribution of one of these parts shall be positive, and the other negative, and so that the total solid angle shall be the algebraic sum of the solid angles subtended by these two parts. If this surface is represented parametrically by the equations

$$\rho = \mathrm{P}(u, v), \qquad \theta = \Theta(u, v), \qquad \phi = \Phi(u, v),$$

under suitable restrictions as to continuity, it will be observed that the Jacobian

$$\frac{D(\theta, \phi)}{D(u, v)}$$

will be positive throughout one of these parts of the cylinder, and negative throughout the other. If now in the double integral above we replace θ, ϕ by the new variables u, v we obtain the integral

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)} du \, dv,^*$$

extended over the total cylindrical surface. In this form the Jacobian takes care of the sign, and thus yields in one integral the result desired. Guided by this illustration we shall proceed to formulate a general definition of a solid angle.

Given any oriented bilateral surface R referred to a system of rectangular coordinates, and represented by one or more sets of equations of the form

$$x = X(u, v), \qquad y = Y(u, v) \qquad z = Z(u, v),$$

where the parameters are so chosen as to satisfy the requirements of Art. 19.

* See, for example, Goursat, *Cours d'Analyse*, Vol. I, §128.

Let $O(x_0, y_0, z_0)$ be any fixed point not on the surface. Change to a new system of rectangular axes with O as origin by a transformation having a positive determinant. Then change to a system of spherical coordinates having O as origin and defined by the equations given above. R can now be represented by one or more sets of equations of the form

$$\rho = \mathrm{P}(u, v), \qquad \phi = \Phi(u, v), \qquad \theta = \Theta(u, v).$$

We shall at first require that the surface shall not be pierced by the polar axis. Later we shall remove this restriction. We define the *solid angle* subtended by R at O to be the integral

$$\int\!\!\int_{R'} \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv, \tag{A}$$

where R' is the totality of the regions in the uv-planes corresponding to R, and where $du\, dv$ is essentially positive. It follows that the solid angle is invariant of any change of parameter made by a transformation having a positive Jacobian at every point. In particular it is immaterial whether the surface is represented by one or many sets of parametric equations.

If the surface has a point $P_1(x_1, y_1, z_1, \rho_1)$ on the polar axis, the integrand is not defined at that point as the Jacobian may become infinite. Let a point P approach P_1. Then it can be shown that

$$\lim \sin\phi \frac{D(\theta, \phi)}{D(u, v)} = \frac{1}{\rho_1^2} \frac{D(x_1, y_1)}{D(u, v)},$$

which is finite and defined. The integrand at such a point shall now be defined to be that limit. Then the integral is a fully defined proper integral. We now define the solid angle in all cases to be the integral (A).

Any rotation of the system of spherical axes about O is accomplished by introducing θ' and ϕ' in place of θ and ϕ by a transformation having a positive Jacobian. By simply making this substitution it can be shown that the integrand, and hence the solid angle, is invariant of this transformation. It follows that the solid angle is independent of the particular choice of polar axis, and is invariant of any change in the original system of rectangular axes made by a transformation with positive Jacobian.

21. *Solid Angle in Terms of a Line Integral.* We shall now express the solid angle by means of the line integral $\int \cos\phi\, d\theta$ taken around the boundary of the

surface. The possibility of doing this is suggested by analogy with Green's theorem in the plane. In fact, in the simpler cases this can be accomplished by a direct application of Green's theorem. From Art. 19 it follows that, if the surface is divided into parts, this integral extended along the entire boundary of the surface is equal to the sum of the integrals of the same function extended along the boundaries of the parts, taken in the positive sense of the curve in each case. This follows since along the common boundary of any two of the parts the integral is extended once in one sense and once in the opposite, and these two integrals cancel each other. We need the following theorem:

THEOREM. *Given any smooth surface, and any point O not on the surface; then it is possible to draw through O a straight line not tangent to the surface, making an arbitrarily small angle with a given line.*

Proof. The condition that a line through O is tangent to the surface, represented by spherical coordinates, is that $D(\theta, \phi)/D(u, v) = 0$ at the point of contact, excluding from consideration points of the surface in the neighborhood of the polar axis. This will be no limitation on the generality of the proof, as the polar axis may be changed if desired. Now divide the surface into a finite number of parts each of which can be represented by an equation of at least one of the following forms:

$$(1)\ \rho = \lambda(\theta, \phi), \quad \text{or} \quad (2)\ \theta = \mu(\phi, \rho), \quad \text{or} \quad (3)\ \phi = \nu(\rho, \theta),$$

where λ, μ, ν, and their first derivatives are single valued and continuous. That this is possible follows directly from Art. 17. The condition that a part can be represented in the first form is that $D(\theta, \phi)/D(u, v) \neq 0$ in the part. Hence no line through O can be tangent at a point of a part of the first class. Discard all parts which can be represented in the first form. Then if the parts are taken sufficiently small, $D(\theta, \phi)/D(u, v) < \varepsilon$ throughout the remainder of the surface, which we will denote by S^-, where ε is an arbitrarily preassigned positive number. Then if α is the solid angle subtended by S^- at O,

$$\begin{aligned} \alpha &= \int\int_{S^-} \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \\ &\leqq \int\int_{S^-} \left| \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \right| \\ &\leqq \int\int_{S^-} \left| \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv \right| \\ &\leqq \varepsilon \int\int_{S^-} du\, dv \quad \leqq \varepsilon \int\int_{S} du\, dv = \varepsilon K, \end{aligned}$$

where K is independent of the method of division. Hence by taking the parts sufficiently small the solid angle subtended by S^- can be made less than that subtended by a preassigned region of the surface of a sphere with O as center. Then it is possible to draw a straight line through O piercing this region and not touching S^-. This proves the theorem.

Such a line meets the surface in at most a finite number of points. Let the polar axis be so chosen. Now divide the surface into a finite number of arbitrarily small parts by Art. 17. At most a definite number independent of the size of the parts, contain points on the polar axis. If the parts are taken sufficiently small the contribution of these parts to the solid angle can be made arbitrarily small. Each of the remaining parts can be represented in at least one of the three following forms:

$$(1)\ \rho = f(\theta, \phi), \qquad (2)\ \phi = f(\rho, \theta), \qquad (3)\ \theta = f(\phi, \rho).$$

We shall show first that

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)} du\, dv = -\int \cos\phi\, d\theta \tag{B}$$

in each part which can be represented in the first form. The line integral is to be extended around the boundary of each part in the positive sense. The condition that a part can be so represented is that $D(\theta, \phi)/D(u, v) \neq 0$, and hence has a constant sign in the part. Hence

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)} du\, dv = \int\int \sin\phi\, d\theta\, d\phi, \tag{C}$$

where $d\theta\, d\phi$ has a constant sign, the same as $D(\theta, \phi)/D(u, v)$. We may think of the transformation $\theta = \theta(u, v)$, $\phi = \phi(u, v)$, as a transformation from the uv plane to the $\theta\phi$ plane. Suppose $D(\theta, \phi)/D(u, v) < 0$, and hence $d\theta\, d\phi < 0$. Apply Green's theorem. We obtain

$$\int\int \sin\phi\, d\theta\, d\phi = \int \cos\phi\, d\theta$$

extended along the boundary of the region in the $\theta\phi$ plane in the positive sense, or

$$-\int \cos\phi\, d\theta$$

extended in the negative sense. Since $D(\theta, \phi)/D(u, v) < 0$, by Art. 18, Th. II c, this corresponds to the positive sense of the curve in the uv plane, and by Art. 19, this corresponds to the positive sense of the boundary of the region on the surface. If $D(\theta, \phi)/D(u, v) > 0$ similar reasoning leads to the same result. Hence the solid angle is given by the integral

$$-\int \cos\phi\, d\theta$$

taken in the positive sense, for every region of the first class.

Consider those parts which can be represented in the form $\phi = f(\rho, \theta)$. Then

$$\int\int \sin\phi \frac{D(\theta, \phi)}{D(u, v)}\, du\, dv = \int\int \sin\phi \frac{\partial\phi}{\partial\rho}\, d\rho\, d\theta.$$

By reasoning similar to that just used the same result is obtained.

We shall adopt a different method for the third case. Let R be any one of the regions already discussed, and C its boundary. Let it be referred to any two systems of spherical coordinates (ρ, θ, ϕ) and (ρ, θ', ϕ') having the same origin and such that no point of R or C is a point of either polar axis. If R is sufficiently small it is possible to construct a conical surface having C as directrix and a point V as vertex so chosen that no element intersects either polar axis. Consider that part of this surface which lies between V and C. Let this be divided into arbitrarily small regions. Then each of these regions can be represented by an equation of at least one of the following forms:

$$\rho = f(\theta, \phi), \qquad \text{or} \qquad \phi = f(\rho, \theta),$$

and also by an equation of at least one of the following forms:

$$\rho = f_1(\theta', \phi'), \qquad \text{or} \qquad \phi' = f_1(\rho, \theta').$$

Hence by each system of coordinates the solid angle is equal to the line integral along C. But the solid angle is invariant of the system of axes. Hence

$$\int_C \cos\phi\, d\theta = \int_C \cos\phi'\, d\theta'.$$

Hence in computing the value of $\int \cos\phi\, d\theta$ around the boundaries of all the

regions of the given surface we may choose the axes arbitrarily for each region. But the axes can always be chosen so a given small region can be represented in one of the forms

$$\rho = f(\theta, \phi) \qquad \text{or} \qquad \phi = f(\rho, \theta).$$

Hence for any sufficiently small region not containing a point of the polar axis the solid angle equals $-\int \cos\phi \, d\theta$ extended along its boundary in the positive sense.

22. *Order of a Point.* Now suppose that the surface is closed. We shall show that the solid angle is a multiple, positive, negative or zero, of 4π. Let the curves q_i be the boundaries of those parts, finite in number, and each arbitrarily small, which have a point in common with the polar axis. These parts may be so chosen that the point in common with the polar axis is an interior point of the part. Let C^- be the total boundaries of the remaining parts. Then since the integral is extended along every boundary once in one sense and once in the opposite,

$$\int_{C^-} \cos\phi \, d\theta + \sum \int_{q_i} \cos\phi \, d\theta = 0.$$

But by taking the parts sufficiently small

$$-\int_{C^-} \cos\phi \, d\theta$$

and hence its equal

$$\sum \int_{q_i} \cos\phi \, d\theta$$

can be made arbitrarily near to the solid angle. But at the same time the latter sum can be made arbitrarily near to

$$\sum \eta_i \int_{q_i} d\theta$$

where $\eta_i = \pm 1$. Hence the solid angle equals

$$\sum \eta_i \int_{q_i} d\theta.$$

On the other hand

$$\int_{C^-} d\theta + \sum \int_{q_i} d\theta = 0,$$

since the integral is extended along every segment once in one sense and once in the opposite. But

$$\int_{C^-} d\theta = 0.$$

This may be shown by taking the parts sufficiently small, and showing that the total variation of θ in any one part is less than 2π, and hence equal to zero. Hence

$$\sum \int_{q_i} d\theta = 0.$$

But

$$\int_{q_i} d\theta = 2 n_i \pi,$$

where n_i is an integer, positive, negative or zero, and the solid angle $\sum \eta_i \int_{q_i} d\theta$ differs from $\sum \int_{q_i} d\theta$ by twice the contribution of those integrals for which η_i is negative. Hence *the solid angle equals* $4 n \pi$, *where n is an integer, positive, negative or zero.* Then define the *order* of the point O with respect to the surface to be the number n. The order of a point on the surface is not defined.

The solid angle is invariant of the particular choice of the polar axes, and of any change of the parameters u, v in any part of the surface, provided the transformation has a positive Jacobian. If the parameters are changed at all points of the surface by transformations having negative Jacobians, the sign of the solid angle is changed but its absolute value is invariant. Hence the same statements are true of the order of a point. For a like reason the order of a point is invariant of any change in the rectangular axes to which the surface is referred, effected by a transformation with positive Jacobian.

THEOREM I. *If the order of a point O not on the surface is n, then all points in the neighborhood of O are of order n.*

Proof. The integral defining the solid angle is the integral over a finite region of a uniformly continuous function of all the variables involved, including the coordinates of O. Hence the solid angle, and therefore the order of O is continuous when O is not on the surface. But the order can vary only by a multiple of unity. Hence it is constant.

Corollary. *The points of the order of a given point form one or more continua.*

THEOREM II. *If two points are of different orders with respect to a given surface, then any simple curve joining them has a point in common with the surface.*

The proof is the same as in two dimensions (Art. 8).

III. THE DIVISION OF SPACE BY A CLOSED SURFACE.

23. The theorem that a closed bilateral surface which satisfies Condition B (Art. 17) divides space into two continua is proved by the aid of two lemmas entirely analogous to those used in two dimensions.

FIRST LEMMA. *If P_0 is a point of a closed bilateral surface which satisfies Condition B, then near P_0 there are two points whose orders with respect to the given surface differ by unity.*

Proof. Let the surface be arbitrarily oriented. If the surface is not smooth at P_0, there is a point of the surface near P_0 at which it is smooth, and which may be used instead of P_0. Hence we may assume that the surface is smooth at P_0. Transform to a new set of rectangular coordinates with origin at P_0, and so chosen that the surface near P_0 can be represented by one equation $z = f(x, y)$, where f is single valued and continuous. The axes can at the same time be so chosen that the z-axis has only a finite number of points in common with the surface (Art. 21, Theorem). It is possible to choose two points $O^+(o, o, \delta)$ and $O^-(o, o, -\delta)$ where δ is so chosen that no point of the surface except P_0 lies on the segment $O^- O^+$ of the z-axis. Now refer to two systems of spherical coordinates having the origin at O^+ and O^- respectively, and the positive z-axis as positive polar axis. Cut out from the surface small regions, each containing one of the points common to the surface and the polar axis. In each case the sum of the integrals $\int \cos \phi \, d\theta$, extended around these regions in the positive sense, is arbitrarily near to the solid angle subtended by the surface at the origin (Art. 21). The contribution of the part containing P_0 to the angle at O^+ differs from the contribution of the same part to the angle at O^- by a number arbitrarily near to 4π. That of the remaining parts in the two cases differ by an arbitrarily

small number. But the orders of O^+ and O^- can differ only by integers, hence they differ by unity.

SECOND LEMMA. *Given any three dimensional continuum R, and a surface S:*

$$z = f(x, y), \quad \text{or} \quad y = f(z, x), \quad \text{or} \quad x = f(y, z),$$

where f is single valued and continuous:

(*a*) *If R contains all points of S except possibly its boundary points which may lie in the boundary of R, then the totality R^- of points of R not on S form at most two continua;*

(*b*) *If also S has a simple regular boundary one point of which is in R, then R^- forms one continuum.*

Proof. (*a*) Suppose S can be represented by the equation $z = f(x, y)$. The other cases are similar. (See Fig. 6, in which the surface S is represented

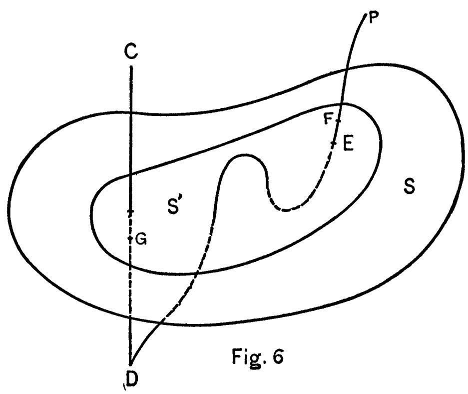

Fig. 6

but not the boundary of R). Draw a straight line CD parallel to the z-axis, lying wholly in R, and bisected at a point of the surface S, and such that C is above the surface S. Let P be any point of R^- which cannot be joined to D by a simple curve wholly in R^-. If there is no such point the theorem is granted. Otherwise join P to D by a simple curve PD wholly in R. This curve will have a point in common with the surface S. Let PE be an arc of PD having one end E on the surface S, but containing no other point of S. Choose a region S' of S whose interior and boundary lies wholly in R, and containing E and the point

common to CD and the surface S. Define two assemblages N^+ and N^- analogous to that of Art. 6, Example 3, as follows:

$$\begin{array}{ll} z = f(x, y) + r, & (x, y) \text{ in } S', \\ 0 < r < h & \text{for } N^+, \\ -h < r < 0 & \text{for } N^-. \end{array}$$

These can be proved to be continua in a manner analogous to that just referred to. Choose a point F on the arc PE, and so near to E that it lies either in N^+ or N^-. Suppose it lay in N^-. Choose a point G on CD in N^-. Then F and G can be joined by a simple curve wholly in N^-. Hence the simple curve $PFGD$ lies wholly in R^-, which is contrary to hypothesis. Hence F must lie in N^+, and by similar reasoning P can be joined to C by a simple curve wholly in R^-. Hence the points of R^- form at most two continua.

(b) Suppose the surface is represented as in the first case, but let it be bounded by a simple regular curve having a point P_0 interior to R. If this point is a vertex there is a point of the boundary of S near it which is not a vertex, and which lies in R. Hence we may assume that it is not a vertex. Let the surface now be extended slightly past P_0. By reasoning similar to that of the first case it can be proved that the points of R not on this enlarged surface form at most two continua. If they form one continuum the theorem is granted. If they form two continua, they can be annexed to each other by the adjunction of the points added to the given surface, thus forming one continuum.

MAIN THEOREM. *The points of space not on a given simple closed bilateral surface which satisfies Condition B form two continua, of each of which the entire surface is the total boundary.*

Proof. In the neighborhood of any point of the surface there are two points of different orders with respect to the surface (First Lemma). Hence the points of space not on the surface form *at least two* continua (Art. 22, Th. II). Divide the surface into parts each of which can be represented in at least one of the following forms:

$$x = f(y, z), \qquad \text{or} \qquad y = f(z, x), \qquad \text{or} \qquad z = f(x, y).$$

Discard these, one at a time in such an order that each part after the first when discarded shall have a portion at least of its boundary in common with a part

already discarded. Then replace them in reverse order. By the second part of the Second Lemma each of these except the last replaced does not divide the region consisting of all space less the points already cut out. By the first part of the same lemma the last part replaced divides the resulting region into *at most two* continua. Hence the points of space not on the surface form just two continua.

Any point of the surface is a boundary point of each continuum (First Lemma, and Art. 6). Any point not on the surface belongs to one of the continua and hence is not a boundary point. This proves the theorem.

A discussion of interior, exterior and normals might be made analogous to that in two dimensions. More general surfaces, having edges or vertices may be defined in a manner analogous to the definition of a smooth surface given in Art. 16. If such a surface satisfies *Condition B* (Art. 17), then the foregoing discussion applies to it.